

EXTREME VALUES OF $\zeta'(\rho)$

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ABSTRACT. In this article we exhibit small and large values of $\zeta'(\rho)$ by applying Soundararajan's resonance method. Our results assume the Riemann hypothesis.

1. INTRODUCTION

Let $\zeta(s)$ denote the Riemann zeta function and let ρ denote a non-trivial zero of this function. A famous conjecture due to Riemann asserts that all non-trivial zeros ρ have real part equal to one-half. This is the Riemann hypothesis. In this article we are concerned with large and small values of $\zeta'(\rho)$. Note that if $|\zeta'(\rho)|$ were small then we would expect a small gap between consecutive zeros of $\zeta(s)$ nearby. An extreme example of this phenomenon is that if ρ is a multiple zero of the zeta function then $\zeta'(\rho) = 0$. On the other hand, if $\zeta'(\rho)$ were large we would expect a large gap between zeros of $\zeta(s)$ nearby. This has been observed numerically in Odlyzko [11]. Also Soundararajan [15] has conjectured that a zero of $\zeta(s)$ close to the half line would correspond to nearby pair of close zeros of the zeta function on the half-line. Recall that the phenomenon of a close pair of zeros of $\zeta(s)$ is referred to as Lehmer's phenomenon. One reason for our interest in such small spaces between the zeros of zeta is due to their connection to the non-existence of Landau-Siegel zeros. This connection was first noticed by Montgomery in [6] and Montgomery and Weinberger in [8]. This idea was further explored by Conrey and Iwaniec in [3]. The problem of the true size of $\zeta'(\rho)$ remains an open question. Under the Riemann hypothesis, we have by an argument of Littlewood, that there exists $c_0 > 0$ such that

$$|\zeta'(\rho)| \ll \exp\left(\frac{c_0 \log |\gamma|}{\log \log |\gamma|}\right)$$

where $\gamma = \text{Im}(\rho)$. This last notation shall be employed throughout the article. On the other hand, we are also interested in small values of $|\zeta'(\rho)|$. Consider $\Theta = \inf\{c \mid |\zeta'(\rho)|^{-1} \ll \gamma^c\}$ defined by Gonek [4] in his study of $M(x)$, the summatory function of the Möbius function. Since the Riemann hypothesis implies $|\zeta'(\rho)| \ll |\rho|^\epsilon$ one expects that $\Theta \geq 0$. On the other hand, the GUE conjecture which asserts that the distribution of consecutive zeros of the zeta function obey the GUE distribution suggests that $\Theta = \frac{1}{3}$ and hence we should have

$$|\zeta'(\rho)| \ll \gamma^{-1/3+\epsilon}$$

infinitely often.

In this article we shall produce results exhibiting both large and small values of $|\zeta'(\rho)|$. These results are obtained by a novel idea due to Soundararajan [16]. The method, coined the resonance method, will be explained shortly. We begin with the large values result.

Theorem 1. *Assume the Riemann hypothesis. For each $A > 0$ we have*

$$|\zeta'(\rho)| \gg_A (\log |\gamma|)^A$$

for infinitely many γ .

I would like to note that Soundararajan has informed me that he has proven that

$$\sum_{0 < \gamma < T} |\zeta'(\rho)|^{2k} \gg_k T(\log T)^{(k+1)^2} \quad (1)$$

by the lower bound method of Rudnick and Soundararajan [12], [13]. Clearly, (1) implies Theorem 1. However, as this remains unpublished, we present our proof of Theorem 1. Thus under the Riemann hypothesis, the lower bound method [12], [13] can give omega results for $\zeta'(\rho)$ of the same strength as the resonance method [16]. This stems from the fact that we are unable to evaluate a certain weighted sum of $\zeta'(\rho)$ without making assumptions about the zeros of Dirichlet L -functions (see Proposition 4 parts (ii) and (iii) that follow). If we are willing to assume an additional hypothesis concerning the location of the zeros of Dirichlet L -functions we can improve Theorem 1 significantly and we can obtain a result of the same quality as Soundararajan's results [16]. We shall require the following:

Large zero-free region conjecture. There exists a positive constant c'_0 sufficiently large such that for each $q \geq 1$ and each character χ modulo q the Dirichlet L -function $L(s, \chi)$ does not vanish in the region

$$\sigma \geq 1 - \frac{c'_0}{\log \log(q(|t| + 4))}$$

where $s = \sigma + it$.

Note that this conjecture is significantly weaker than the generalized Riemann hypothesis. However, it is a sufficiently strong hypothesis to rule out the existence of Siegel zeros. Recall that the classical zero-free region for Dirichlet L -functions is $L(s, \chi)$ does not vanish in the region

$$\sigma \geq 1 - \frac{c_1}{\log(q(|t| + 3))}$$

for some $c_1 > 0$ with the possible exception of one simple real zero in the case χ is quadratic.

Theorem 2. *Assume the Riemann hypothesis and the large zero-free region conjecture. There are arbitrarily large values of γ such that*

$$|\zeta'(\rho)| \gg \exp \left(c_2 \sqrt{\frac{\log |\gamma|}{\log \log |\gamma|}} \right)$$

where $c_2 = \frac{1}{\sqrt{2}} - \epsilon$ is valid.

We also prove a result for small values of $|\zeta'(\rho)|$. Surprisingly, this proof is significantly easier than the proof of Theorem 2.

Theorem 3. *Assume the Riemann hypothesis. We have*

$$|\zeta'(\rho)| \ll \exp \left(-c_3 \sqrt{\frac{\log |\gamma|}{\log \log |\gamma|}} \right)$$

for infinitely many γ where $c_3 = \sqrt{\frac{2}{3}} - \epsilon$ is valid.

2. NOTATION

We shall use Vinogradov's notation $f(x) \ll g(x)$ to mean there exists a $C > 0$ such that $|f(x)| \leq Cg(x)$ for all x sufficiently large. We denote $f(x) = O(g(x))$ to mean the same thing. Also, $f(x) = o(g(x))$ means $f(x)/g(x) \rightarrow 0$ as $x \rightarrow \infty$. We shall consider arbitrary sequences x_n supported on an interval $[1, M]$ and we employ the notation

$$\|x_n\|_\infty = \max_{n \leq M} |x_n| \text{ and } \|x_n\|_p = \left(\sum_{n \leq M} |x_n|^p \right)^{1/p}.$$

We now define some basic arithmetic functions. We define $\mu(n)$, the Mobius function, to be the coefficient of n^{-s} in the Dirichlet series $\zeta(s)^{-1} = \sum_{n=1}^{\infty} \mu(n)n^{-s}$. We define $\Lambda_k(n)$ to be the coefficient of n^{-s} in the Dirichlet series of $(-1)^k \zeta^{(k)}(s)/\zeta(s)$. Another way to express this is $\Lambda_k(n) = (\mu * \log^k)(n)$. Note that $\Lambda_k(n)$ is supported on those integers with at most k distinct prime factors. We define $\tau_k(n)$, the k -th divisor function, to be the coefficient of n^{-s} in the Dirichlet series $\zeta(s)^k = \sum_{n=1}^{\infty} \tau_k(n)n^{-s}$.

3. EXPLANATION OF THE RESONANCE METHOD

In this section we outline the resonance method. Soundararajan [16] recently invented this simple method to find large values of $|\zeta(1/2 + it)|$ (and also other L -functions and character sums). Under the Riemann hypothesis it is known that

$$|\zeta(1/2 + it)| \ll \exp \left(\frac{c'_1 \log |t|}{\log \log |t|} \right)$$

where $c'_1 > 0$ is explicitly given. However, it has been proven by Montgomery [7], assuming the Riemann hypothesis, that there exist arbitrarily large t such that

$$|\zeta(1/2 + it)| \gg \exp \left(c'_2 \sqrt{\frac{\log |t|}{\log \log |t|}} \right)$$

for some positive constant c'_2 . Later, Balasubramanian and Ramachandra [1] gave an unconditional proof of this result with an explicit value $c'_2 < 1$. The new method permits the choice $c'_2 = 1 - \epsilon$. We now sketch the method. Consider the mean values

$$\int_T^{2T} \zeta(1/2 + it) |A(it)|^2 dt \text{ and } \int_T^{2T} |A(it)|^2 dt$$

where $A(s) = \sum_{n \leq M} x_n n^{-s}$ is a Dirichlet polynomial with arbitrary positive coefficients x_n and $y \leq T^{1-\epsilon}$. A standard calculation shows that

$$\frac{\int_T^{2T} \zeta(1/2 + it) |A(it)|^2 dt}{\int_T^{2T} |A(it)|^2 dt} = \left(\frac{\sum_{nu \leq M} \frac{x_n x_{nu}}{\sqrt{u}}}{\sum_{n \leq M} x_n^2} \right) (1 + o(1)).$$

By taking absolute values we deduce that

$$\max_{T \leq t \leq 2T} |\zeta(1/2 + it)| \geq \left(\frac{\sum_{nu \leq M} \frac{x_n x_{nu}}{\sqrt{u}}}{\sum_{n \leq M} x_n^2} \right) (1 + o(1)) . \quad (2)$$

The problem is thus reduced to optimizing the fraction on the right. Soundararajan [16] shows that the maximum of the above quotient is

$$\exp \left(\sqrt{\frac{\log M}{\log \log M}} (1 + o(1)) \right)$$

and this is obtained by choosing $x_n = f(n)$ where $f(n)$ is multiplicative and supported on squarefree numbers. We define f on the primes as follows: let $L = \sqrt{\log M \log \log M}$ and set

$$f(p) = \begin{cases} \frac{L}{\sqrt{p} \log p} & \text{if } L^2 \leq p \leq \exp((\log L)^2) \\ 0 & \text{else} \end{cases} .$$

The strategy of this article is to follow the above argument. We require asymptotic formulae for the mean values

$$S_1 = \sum_{0 < \gamma < T} \zeta'(\rho) A(\rho) A(1 - \rho) \text{ and } S_2 = \sum_{0 < \gamma < T} A(\rho) A(1 - \rho)$$

where $A(s) = \sum_{n \leq M} x_n n^{-s}$ has arbitrary real coefficients x_n , $y = T^\theta$, and $\theta < 1/2$. Observe that if the Riemann Hypothesis is true then $|A(\rho)|^2 = A(\rho) A(1 - \rho)$ and thus

$$S_1 = \sum_{0 < \gamma < T} \zeta'(\rho) |A(\rho)|^2 \text{ and } S_2 = \sum_{0 < \gamma < T} |A(\rho)|^2 . \quad (3)$$

In fact, we shall show that S_1/S_2 is essentially the same quotient of quadratic forms as in (2).

We have the following formulae for S_1 and S_2 :

Proposition 4. (i) Suppose that $|x_n| \ll T^\epsilon$ and $\theta < 1$. Then we have

$$S_2 = N(T) \sum_{m \leq M} \frac{x_m^2}{m} - \frac{T}{\pi} \sum_{m \leq M} \frac{(\Lambda * x)(m) x_m}{m} + o(T) \quad (4)$$

where $N(T)$ is the number of zeros of the zeta function in the box $0 \leq \operatorname{Re}(s) \leq 1$, $0 \leq \operatorname{Im}(s) \leq T$.

(ii) Suppose that $|x_n| \ll \tau_r(n)(\log T)^C$ for some $C > 0$ and $\theta < 1/2$. Then we have

$$S_1 = \frac{T}{2\pi} \left(\sum_{nu \leq M} \frac{x_u x_{nu} r_0(n)}{nu} + \sum_{\substack{a, b \leq M \\ (a, b) = 1}} \frac{r_1(a, b)}{ab} \sum_{g \leq \min(\frac{M}{a}, \frac{M}{b})} \frac{x_{ag} x_{bg}}{g} \right) + o(T) \quad (5)$$

where

$$\begin{aligned} r_0(n) &= \frac{1}{2} P_2(\log(\frac{T}{2\pi})) - P_1(\log(\frac{T}{2\pi})) \log n - \frac{1}{2} (\log n)^2 + (\Lambda * \log)(n) , \\ r_1(a, b) &= \frac{1}{2} \Lambda_2(a) - R_1(\log(\frac{T}{b})) \Lambda(a) - \tilde{R}_1(\log(\frac{T}{b})) \alpha_1(a) - \alpha_2(a) , \end{aligned}$$

$P_2, P_1, R_1, \tilde{R}_1$ are monic polynomials of degrees 2, 1, 1, 1 respectively. α_2, α_1 are arithmetic functions. α_2 is supported on a with $\omega(a) \leq 2$ and α_1 is supported on prime powers. Moreover, $\alpha_1(p^j) \ll \frac{\log p}{p}$, $\alpha_2(p^j) \ll \frac{j(\log p)^2}{p}$, and $\alpha_2(p^j q^k) \ll (\log p)(\log q)(p^{-1} + q^{-1})$.

(iii) Assume the large zero-free region conjecture. The formula for S_1 in (ii) remains valid under the assumption that $x_n = \sqrt{n}f(n)$ and $\theta < 1/3$.

Proof. The proofs of (ii) and (iii) may be found in Theorem 1.3 of [10]. The formula for S_2 in (i) is mentioned without proof on page 6 of [2]. It can be proven by following the argument of [9] Lemma 3. \square

From Proposition 4, we can explain our strategy for proving Theorem 2. We shall show that in the formulae (5) and (4) for S_1 and S_2 the significant terms are

$$\frac{T \log^2(\frac{T}{2\pi})}{4\pi} \sum_{nu \leq M} \frac{x_u x_{nu}}{nu} \text{ and } \frac{T \log(\frac{T}{2\pi})}{2\pi} \sum_{m \leq M} \frac{x_m^2}{m}$$

respectively. By choosing $x_n = \sqrt{n}f(n)$ we see that

$$\max_{T \leq \gamma \leq 2T} |\zeta'(\rho)| \geq \frac{S_1}{S_2} \approx \frac{\log(\frac{T}{2\pi})}{2} \left(\frac{\sum_{rn \leq M} \frac{f(n)f(nr)}{\sqrt{r}}}{\sum_{n \leq M} f(n)^2} \right) = \exp \left(\sqrt{\frac{\log M}{\log \log M}} (1 + o(1)) \right).$$

This is the essential content of Theorem 2. In order to make this argument rigorous, we will show that each of the other terms in the formulae for S_1 and S_2 are smaller than the principal terms. The argument for Theorem 3 is very similar. In this case we consider

$$S_3 = \sum_{T < \gamma < 2T} \zeta'(\rho)^{-1} |A(\rho)|^2 \text{ and } S_2 = \sum_{T < \gamma < 2T} |A(\rho)|^2.$$

As before we will show that the ratio S_3/S_2 gives rise to the same quadratic form as in (2).

4. LARGE VALUES OF $\zeta'(\rho)$: PROOF OF THEOREM 1

In this section we prove Theorem 1. As explained previously our strategy is to evaluate asymptotically S_1/S_2 for a certain choice of coefficients. As we are only assuming the Riemann hypothesis, we are restricted to choosing $x_n = \tau_r(n)$ with $r \in \mathbb{N}$. In the course of this calculation, we shall encounter several other multiplicative functions. We define

$$f_1(n) = \prod_{p^e || n} \frac{\sum_{j=0}^{\infty} \frac{\tau_r(p^{e+j})}{p^j}}{\sum_{j=0}^{\infty} \frac{\tau_r(p^j)}{p^j}} \text{ and } f_2(n) = \prod_{p^e || n} \frac{\sum_{j=0}^{\infty} \frac{\tau_r(p^{e+j})\tau_r(p^j)}{p^j}}{\sum_{j=0}^{\infty} \frac{\tau_r(p^j)^2}{p^j}}.$$

Note that for $i = 1, 2$ $f_i(p) = r(1 + O(p^{-1}))$. The asymptotic evaluation of S_1 will require the evaluation of several sums of standard arithmetic functions. We shall employ the following:

Lemma 5. Let $a, b, k, r, u \in \mathbb{N}$.

(i)

$$\sum_{n \leq x} \tau_r(nu) = \frac{f_1(u)x(\log x)^{r-1}}{(r-1)!} (1 + O((\log x)^{-1})).$$

(ii)

$$\sum_{n \leq x} \tau_r(n)f_1(n) = \frac{C_0 x(\log x)^{r^2-1}}{(r^2-1)!} (1 + O((\log x)^{-1}))$$

where

$$C_0 = \prod_p \left(1 - \frac{1}{p}\right)^r \sum_{j=0}^{\infty} \frac{\tau_r(p^j) f_1(p^j)}{p^j} = \prod_p \left(1 - \frac{1}{p}\right)^{r^2+r} \sum_{j=0}^{\infty} \frac{\tau_r(p^j) \tau_{r+1}(p^j)}{p^j}. \quad (6)$$

(iii)

$$\sum_{n \leq x} f_2(n) = \frac{C_1 x (\log x)^{r-1}}{(r-1)!} (1 + O((\log x)^{-1}))$$

where

$$C_1 = \prod_p (1 - 1/p)^r \sum_{j=0}^{\infty} \frac{f_2(p^j)}{p^j} = \prod_p (1 - 1/p)^r \frac{\sum_{j=0}^{\infty} \frac{\tau_r(p^j) \tau_{r+1}(p^j)}{p^j}}{\sum_{j=0}^{\infty} \frac{\tau_r(p^j)^2}{p^j}}. \quad (7)$$

(iv)

$$\sum_{n \leq x} \frac{\tau_r(an) \tau_r(bn)}{n} = C_2 f_2(a) f_2(b) \frac{(\log x)^{r^2}}{(r^2)!} (1 + O((\log x)^{-1}))$$

where

$$C_2 = \prod_p (1 - 1/p)^{r^2} \sum_{j=0}^{\infty} \frac{\tau_r(p^k)^2}{p^k}.$$

Notice that it follows immediately from (6) and (7) that $C_0 = C_1 C_2$.

(v)

$$\sum_{n \leq x} \Lambda_k(n) = kx (\log x)^{k-1} (1 + O((\log x)^{-1})).$$

(vi) For $i = 1, 2$

$$\sum_{n \leq x} \Lambda(n) f_i(n) = rx (1 + O((\log x)^{-1})).$$

(vii) For $i = 1, 2$

$$\sum_{n \leq x} \Lambda_2(n) f_i(n) = (r^2 + r)x (\log x) (1 + O((\log x)^{-1})).$$

Proof. Since the proofs of (i) – (iv) are very similar we shall just prove part (iv). We give a sketch of the proof as the argument is standard (see for example [14]). We define the Dirichlet series $H(s) = \sum_{n=1}^{\infty} \tau_r(an) \tau_r(bn) n^{-s}$ and since τ_r is multiplicative we have the factorization

$$H(s) = \prod_{(p,ab)=1} \left(\sum_{k=0}^{\infty} \frac{\tau_r(p^k)^2}{p^{ks}} \right) \prod_{p^e || a} \sum_{k=0}^{\infty} \frac{\tau_r(p^{e+k}) \tau_r(p^k)}{p^{ks}} \prod_{p^f || b} \sum_{k=0}^{\infty} \frac{\tau_r(p^k) \tau_r(p^{f+k})}{p^{ks}}.$$

Next we define for $s \in \mathbb{C}$ and $n \in \mathbb{N}$

$$F(s; n) = \prod_{p^e || n} \left(\frac{\sum_{k=0}^{\infty} \frac{\tau_r(p^{e+k}) \tau_r(p^k)}{p^{ks}}}{\sum_{k=0}^{\infty} \frac{\tau_r(p^k)^2}{p^{ks}}} \right), \quad G(s) = \prod_p (1 - 1/p^s)^{r^2} \sum_{k=0}^{\infty} \frac{\tau_r(p^k)^2}{p^{ks}}$$

and thus $H(s) = \zeta(s)^{r^2} F(s, ab) G(s)$. Moreover, we notice that $F(1; n) = f_2(n)$ and $G(1) = C_2$. By Perron's formula,

$$\sum_{n \leq x} \frac{\tau_r(an) \tau_r(bn)}{n} = \frac{1}{2\pi i} \int_{\kappa-iU}^{\kappa+iU} H(s+1) \frac{x^s ds}{s} + O\left(\frac{(\log x)^{r^2}}{U} + \frac{1}{x^{1-\epsilon}} \left(1 + x \frac{\log U}{U}\right) \right)$$

with $\kappa = (\log x)^{-1}$. Let $\Gamma(U)$ denote the contour consisting of $s \in \mathbb{C}$ such that $\operatorname{Re}(s) = -\frac{c'}{\log(|\operatorname{Im}(s)|+2)}$ and $|\operatorname{Im}(s)| \leq U$ for an appropriate $c' > 0$. We deform the contour past $\operatorname{Re}(s) = 0$ line to $\Gamma(U)$ picking up the residue at $s = 0$. The residue at $s = 0$ equals

$$C_2 f_2(a) f_2(b) \frac{(\log x)^{r^2}}{(r^2)!} (1 + O((\log x)^{-1}))$$

which corresponds to the main term. Employing standard bounds for $\zeta(s)$ in the zero-free region we can show that contribution of the integral on $\Gamma(U)$ is smaller than the main term for an appropriate choice of U by at least one factor of $\log x$. Part (v) is a well known fact. Part (vi) follows from the fact that Λ is supported on the prime powers and $\Lambda(p^j) = \log(p)$. Part (vii) follows from the fact that Λ_2 is supported on those n with $\omega(n) \leq 2$ and moreover $\Lambda_2(pq) = 2 \log p \log q$, $\Lambda_2(p) = (\log p)^2$, and $f_i(p) = r(1 + O(p^{-1}))$. \square

We are now prepared to prove Theorem 1. In the course of the proof, we will encounter the following integrals:

$$\begin{aligned} i(u, v) &:= \int_0^1 x^u (1-x)^v dx = \frac{u!v!}{(u+v+1)!}, \\ c_X(u, v) &:= \int_1^X \frac{(\log X/t)^u (\log t)^v}{t} dt = (\log X)^{u+v+1} i(u, v) \end{aligned} \quad (8)$$

where $X \geq 1$.

Proof of Theorem 1. By Proposition 4 we may write $S_1 = \tilde{S}_1 + o(T)$ where

$$\tilde{S}_1 = \frac{T}{2\pi} \left(\sum_{nu \leq M} \frac{x_u x_{nu} r_0(n)}{nu} + \sum_{\substack{a, b \leq M \\ (a, b) = 1}} \frac{r_1(a, b)}{ab} \sum_{g \leq \min(\frac{M}{a}, \frac{M}{b})} \frac{x_{ag} x_{bg}}{g} \right)$$

and

$$\begin{aligned} r_0(n) &= \frac{1}{2} P_2(\log(\frac{T}{2\pi})) + \sum_{d|n} g(d), \\ g(d) &= -(P_1(\log(\frac{T}{2\pi})) + \log d) \Lambda(d) + \frac{\Lambda_2(d)}{2}. \end{aligned} \quad (9)$$

Thus we have $\tilde{S}_1 = \frac{T}{2\pi} \left(\frac{P_2(\log(T/2\pi))}{2} T_1 + T_2 + T_3 \right)$ where

$$\begin{aligned} T_1 &= \sum_{nu \leq M} \frac{x_u x_{nu}}{nu}, \\ T_2 &= \sum_{dnu \leq M} \frac{g(d) x_u x_{dnu}}{dnu}, \\ T_3 &= \sum_{\substack{a, b \leq M \\ (a, b) = 1}} \frac{r_1(a, b)}{ab} \sum_{g \leq \min(\frac{M}{a}, \frac{M}{b})} \frac{x_{ag} x_{bg}}{g}. \end{aligned} \quad (10)$$

4.1. **Evaluation of T_1 .** Now by Lemma 5 (i) and (ii) we have

$$\begin{aligned} T_1 &= \sum_{u \leq M} \frac{\tau_r(u)}{u} \int_{1^-}^{M/u} t^{-1} d \left(\sum_{n \leq t} \tau_r(nu) \right) \\ &\sim \sum_{u \leq M} \frac{\tau_r(u) f_1(u)}{u} \frac{(\log M/u)^r}{r!} = \frac{1}{r!} \int_{1^-}^M \log(M/t)^r t^{-1} d \left(\sum_{u \leq t} \tau_r(u) f_1(u) \right) \\ &\sim \frac{1}{r!} \int_1^M \frac{(\log(M/t))^r}{t} \frac{C_0 (\log t)^{r^2-1}}{(r^2-1)!} dt. \end{aligned}$$

By (8) it follows that

$$T_1 \sim \frac{C_0}{r!(r^2-1)!} c_M(r, r^2-1) = \frac{C_0 (\log M)^{r^2+r}}{(r^2+r)!}.$$

4.2. **Evaluation of T_2 .** Since the calculation of T_2 and T_3 are rather similar to that of T_1 we shall not record every step of their calculation. By Lemma 5 (i) we have

$$T_2 \sim \sum_{d \leq M} \frac{g(d)}{d} \sum_{u \leq M/d} \frac{\tau_r(u)}{u} \frac{f_1(du) \log(M/du)^r}{r!}.$$

As g is supported on those integers d with $\omega(d) \leq 2$ we have

$$\begin{aligned} T_2 &\sim \sum_{d \leq M} \frac{g(d) f_1(d)}{d} \sum_{u \leq M/d} \frac{\tau_r(u)}{u} \frac{f_1(u) \log(M/du)^r}{r!} \\ &= \frac{1}{r!} \sum_{d \leq M} \frac{g(d) f_1(d)}{d} \int_1^{M/d} \frac{\log(M/dt)^r}{t} \frac{C_0 (\log t)^{r^2-1}}{(r^2-1)!} dt \\ &= \frac{C_0}{(r^2+r)!} \sum_{d \leq M} \frac{g(d) f_1(d)}{d} (\log M/d)^{r^2+r} \end{aligned}$$

where we have invoked Lemma 5 (ii) and (8). By (9), Lemma 5 (vi) and (vii) we obtain

$$\sum_{n \leq x} g(n) f_1(n) \sim x \left(\frac{r^2-r}{2} \log x - r P_1(\log(\frac{T}{2\pi})) \right).$$

From this we deduce

$$\begin{aligned} T_2 &= \frac{C_0}{(r^2+r)!} \int_1^M \frac{(\log M/t)^{r^2+r}}{t} \left(\frac{r^2-r}{2} \log(t) - r P_1(\log(\frac{T}{2\pi})) \right) dt \\ &\sim \frac{C_0}{(r^2+r)!} \left(\frac{r^2-r}{2} c_M(r^2+r, 1) - \frac{r}{\theta} c_M(r^2+r, 0) \right) (\log M)^{r^2+r+2} \end{aligned}$$

and it follows from (8) that

$$T_2 \sim \frac{C_0 (\log M)^{r^2+r+2}}{(r^2+r+2)!} \left(\frac{r^2-r}{2} - \frac{r}{\theta} (r^2+r+2) \right).$$

4.3. **Evaluation of T_3 .** By Lemma 5 (iv) it follows that

$$T_3 \sim \frac{C_2}{(r^2)!} \sum_{\substack{a, b \leq M \\ (a, b)=1}} \frac{r_1(a, b) f_2(a) f_2(b)}{ab} \left(\log \min \left(\frac{M}{a}, \frac{M}{b} \right) \right)^{r^2}$$

where $r_1(a, b)$ is defined by (6). We shall write this last sum as $T'_3 + T''_3$ where T'_3 is the sum over the terms for which $a < b \leq M$ and T''_3 consists of the terms for which $b < a \leq M$. We have

$$T'_3 \sim \frac{C_2}{(r^2)!} \sum_{b \leq M} \frac{f_2(b) \log(M/b)^{r^2}}{b} \sum_{\substack{a < b \\ (a, b)=1}} \frac{(1/2)\Lambda_2(a) - \Lambda(a)R_1(\log(T/b))}{a} f_1(a)$$

since it may be checked that the contribution from the term $-\tilde{R}_1(\log(T/b))\alpha_1(a) - \alpha_2(a)$ is $\ll (\log T)^{r^2+r+1}$. By Lemma 5 (vi) and (vii)

$$\sum_{a \leq x} f_1(a) ((1/2)\Lambda_2(a) - \Lambda(a)R_1(\log(T/b))) \sim \frac{(r^2+r)}{2} x \log x - rR_1(\log(T/b))x$$

and it follows that

$$\begin{aligned} T'_3 &\sim \frac{C_2}{(r^2)!} \sum_{b \leq M} \frac{f_2(b) \log(M/b)^{r^2}}{b} \int_1^b \frac{(1/2)(r^2+r) \log t - rR_1(\log(T/b))}{t} dt \\ &= \frac{C_2}{(r^2)!} \sum_{b \leq M} \frac{f_2(b) \log(M/b)^{r^2}}{b} \left(\frac{r^2+r}{4} (\log b)^2 - rR_1(\log(T/b)) \log b \right). \end{aligned}$$

By Lemma 5 (iii)

$$\begin{aligned} T'_3 &= \frac{C_2}{(r^2)!} \int_1^M \frac{\log(M/t)^{r^2}}{t} \left(\frac{r^2+r}{4} (\log t)^2 - rR_1(\log(T/t)) \log t \right) \frac{C_1}{(r-1)!} (\log t)^{r-1} dt \\ &= \frac{C_0 (\log M)^{r^2+r+2}}{(r^2+r+2)!} \left(\frac{(r^2+5r)(r+1)r}{4} - \frac{r^2(r^2+r+2)}{\theta} \right). \end{aligned}$$

Next, we consider those terms with $b < a \leq M$. We have

$$T''_3 \sim \frac{C_2}{(r^2)!} \sum_{a \leq M} \sum_{\substack{b < a \\ (a, b)=1}} \frac{(1/2)\Lambda_2(a) - \Lambda(a)R_1(\log(T/b))}{a} \frac{f_1(a) f_2(b) \log(M/a)^{r^2}}{b}$$

since we can show, as before, that the contribution from the term $-\tilde{R}_1(\log(T/b))\alpha_1(a) - \alpha_2(a)$ is $\ll (\log T)^{r^2+r+1}$. Since $\sum_{b \leq x} f_2(b) \sim \frac{C_1}{(r-1)!} x (\log x)^{r-1}$, a similar calculation as above yields

$$T''_3 \sim \frac{C_0}{(r^2)!r!} \int_{1-}^M \frac{\log(M/t)^{r^2} (\log t)^r}{t} d\sigma(t)$$

with $\sigma(t) = \sum_{a \leq t} \left((\Lambda_2(a)/2 - \Lambda(a) \log T) + \frac{r}{r+1} \Lambda(a) \log(a) \right) f_2(a)$. By Lemma 5 (vi) and (vii) $\sigma(t) \sim \left(\frac{r^2+r}{2} + \frac{r^2}{r+1} \right) t \log t - rt(\log T)$ and thus

$$\begin{aligned} T_3'' &\sim \frac{C_0}{(r^2)!r!} \left(\left(\frac{r^2+r}{2} + \frac{r^2}{r+1} \right) c_M(r^2, r+1) - r(\log T) c_M(r^2, r) \right) \\ &= \frac{C_0(\log M)^{r^2+r+2}}{(r^2+r+2)!} \left(\left(\frac{(r^2+r)(r+1)}{2} + r^2 \right) - \frac{r(r^2+r+2)}{\theta} \right). \end{aligned}$$

Collecting our results for T_1, T_2 , and $T_3 = T_3' + T_3''$ we have

$$\begin{aligned} S_1 &\sim \frac{C_0 T (\log M)^{r^2+r+2}}{(r^2+r+2)!} \left(\frac{(r^2+r+2)(r^2+r+1)}{\theta^2} + \left(\frac{r(r-1)}{2} - \frac{r(r^2+r+2)}{\theta} \right) \right. \\ &\quad \left. + (r^2+r) \left(\frac{r^2+5r}{4} + \frac{r+1}{2} + \frac{r^2}{r^2+r} \right) - \frac{(r^2+r+2)(r^2+r)}{\theta} \right) \\ &\geq \frac{C_0 T (\log M)^{r^2+r+2}}{(r^2+r+2)!} \frac{r^2+r+2}{\theta^2} (r^2+r+1 - \theta(r^2+2r)) \\ &\gg \frac{r^4 T (\log M)^{r^2+r+2}}{\theta^2 (r^2+r+2)!} \end{aligned}$$

for $0 < \theta < \frac{1}{2}$ and $r \in \mathbb{N}$. On the other hand, we have the simple bound

$$S_2 \leq \frac{T \log(\frac{T}{2\pi})}{2\pi} \sum_{m \leq M} \frac{\tau_r(m)^2}{m} \ll \frac{T}{\theta} (\log M)^{r^2+1}$$

and thus $\max_{T \leq \gamma \leq 2T} |\zeta'(\rho)| \geq \left| \frac{S_1}{S_2} \right| \gg_r (\log M)^{r+1} \gg (\log T)^{r+1}$. \square

5. LARGER VALUES OF $\zeta'(\rho)$: PROOF OF THEOREM 2

In this section we shall evaluate S_1/S_2 for the choice $x_n = \sqrt{n}f(n)$. Before embarking on this task we will require a few results concerning the coefficients $f(n)$. Moreover, we shall encounter several other multiplicative functions. We define g and h to be multiplicative functions supported on the squarefree numbers. Their values at any prime p are given by

$$g(p) = 1 + f(p)^2 \text{ and } h(p) = 1 + f(p)p^{-1/2}.$$

It will also be convenient to introduce the notation

$$\mathcal{Q}_1 = \prod_p \left(1 + f(p)^2 + \frac{f(p)}{\sqrt{p}} \right), \quad \mathcal{Q}_2 = \prod_p (1 + f(p)^2).$$

Lemma 6. (i)

$$\sum_{nu \leq M} \frac{f(u)f(nu)}{\sqrt{n}} = \mathcal{Q}_1(1 + o(1)),$$

(ii)

$$\sum_{n \leq M} f(n)^2 \leq \mathcal{Q}_2,$$

(iii)

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_2} = \exp \left(\sqrt{\frac{\log M}{\log \log M}} (1 + o(1)) \right).$$

(iv) For $i = 1, 2$

$$\sum_{a \leq M} \frac{\Lambda_i(a)f(a)}{\sqrt{a}g(a)} \ll (\log T)^{i/2+\epsilon}.$$

Proof. (i) We denote the sum to be estimated \mathcal{S} . Thus

$$\mathcal{S} = \sum_{n \leq M} \frac{f(n)}{\sqrt{n}} \sum_{\substack{u \leq M/r \\ (n,u)=1}} f(u)^2 = \sum_{n \leq M} \frac{f(n)}{\sqrt{n}} \left(\prod_{(p,n)=1} (1 + f(p)^2) - \sum_{\substack{u > M/n \\ (n,u)=1}} f(u)^2 \right).$$

By Rankin's trick the error term is bounded by

$$\sum_{n \leq M} \frac{f(n)}{\sqrt{n}} \left(\frac{n}{M} \right)^\alpha \sum_{\substack{u=1 \\ (u,n)=1}}^{\infty} f(u)^2 u^\alpha \leq \frac{1}{M^\alpha} \prod_p \left(1 + p^\alpha f(p)^2 + f(p)p^{\alpha-1/2} \right)$$

for any $\alpha > 0$. On the other hand, since f is multiplicative the main term equals

$$\prod_p \left(1 + f(p)^2 + \frac{f(p)}{\sqrt{p}} \right) + O \left(\frac{1}{M^\alpha} \prod_p \left(1 + f(p)^2 + \frac{f(p)p^\alpha}{\sqrt{p}} \right) \right).$$

We deduce

$$\mathcal{S} = \mathcal{Q}_1 + O \left(\frac{1}{M^\alpha} \prod_p \left(1 + p^\alpha f(p)^2 + \frac{f(p)p^\alpha}{\sqrt{p}} \right) \right). \quad (11)$$

However, it is shown in [16] that the ratio of the error term to the main term in (11) is $\ll \exp(-\alpha \frac{\log M}{\log \log M})$ for the choice $\alpha = \frac{1}{(\log L)^3}$. It follows that $\mathcal{S} = \mathcal{Q}_1(1 + o(1))$.

(ii) We have the simple identity

$$\sum_{n \leq M} f(n)^2 \leq \sum_{n \geq 1} f(n)^2 = \mathcal{Q}_2.$$

(iii) Note that

$$\frac{\mathcal{Q}_1}{\mathcal{Q}_2} = \prod_p \left(1 + \frac{f(p)}{\sqrt{p}(1 + f(p)^2)} \right).$$

Taking logarithms of the product we see that

$$\begin{aligned} \log(\mathcal{Q}_1/\mathcal{Q}_2) &= \sum_p \log \left(1 + \frac{f(p)}{\sqrt{p}(1 + f(p)^2)} \right) = \sum_{L^2 \leq p \leq \exp((\log L)^2)} \frac{L}{p \log p(1 + o(1))} \\ &= \frac{L}{\log L^2} (1 + o(1)) = \sqrt{\frac{\log M}{\log \log M}} (1 + o(1)). \end{aligned}$$

(iv) We have

$$\sum_{a \leq M} \frac{\Lambda(a)f(a)}{\sqrt{a}g(a)} = L \sum_{p \leq M} \frac{1}{pg(p)} \ll L \sum_{p \leq M} \frac{1}{p} \ll (\log T)^{1/2+\epsilon}.$$

Note that Λ_2 is supported on integers a satisfying $\omega(a) \leq 2$ and f is supported on squarefree integers. Moreover $\Lambda_2(p) = (\log p)^2$ and $\Lambda_2(pq) = 2 \log p \log q$. From

this, we deduce that

$$\begin{aligned} \sum_{a \leq M} \frac{\Lambda_2(a)f(a)}{\sqrt{a}g(a)} &\ll \sum_{p \leq M} \frac{\Lambda_2(p)f(p)}{\sqrt{p}g(p)} + \sum_{pq \leq M, p \neq q} \frac{\Lambda_2(pq)f(pq)}{\sqrt{pq}g(pq)} \\ &\ll L \sum_{p \leq M} \frac{\log p}{p} + L^2 \left(\sum_{p \leq M} \frac{1}{p} \right)^2 \ll (\log T)^{1+\epsilon}. \end{aligned}$$

□

Proof of Theorem 2. We have from Proposition 4 that

$$\frac{S_1}{S_2} = \frac{\frac{1}{2}P_2(\log(\frac{T}{2\pi}))\Sigma_0 - P_1(\log(\frac{T}{2\pi}))\Sigma_1 - \frac{1}{2}\Sigma_2 + \Sigma_3 + \Sigma_4}{\log(\frac{T}{2\pi})\Sigma_5 - 2\Sigma_6} + o(1)$$

where for $i = 0, 1, 2$

$$\Sigma_i = \sum_{nu \leq M} \frac{f(u)f(nu)(\log n)^i}{\sqrt{n}}$$

and

$$\begin{aligned} \Sigma_3 &= \sum_{nu \leq M} \frac{f(u)f(nu)(\Lambda * \log)(n)}{\sqrt{n}}, \\ \Sigma_4 &= \sum_{\substack{a, b \leq M \\ (a, b) = 1}} \frac{r_1(a, b)}{\sqrt{ab}} \sum_{g \leq \min(\frac{M}{a}, \frac{M}{b})} f(ag)f(bg), \\ \Sigma_5 &= \sum_{m \leq M} f(m)^2, \\ \Sigma_6 &= \sum_{mn \leq M} \frac{\Lambda(n)f(m)f(mn)}{\sqrt{n}}. \end{aligned}$$

By Lemma 6

$$\Sigma_0 = \mathcal{Q}_1(1 + o(1)) \text{ and } \Sigma_5 \leq \mathcal{Q}_2(1 + o(1)). \quad (12)$$

We shall prove the following bounds for the other five sums:

Lemma 7. *We have:*

$$\begin{aligned} \Sigma_1 &\ll \mathcal{Q}_1(\log T)^{1/2+\epsilon}, \quad \Sigma_2, \Sigma_3 \ll \mathcal{Q}_1(\log T)^{1+\epsilon}, \\ \Sigma_4 &\ll \mathcal{Q}_1(\log T)^{3/2+\epsilon}, \quad \Sigma_6 \ll \mathcal{Q}_2(\log T)^{1/2+\epsilon}. \end{aligned}$$

Theorem 2 now easily follows. We deduce from (12) and Lemma 7 that

$$\mathcal{S}_1 = (1/2)\mathcal{Q}_1 \log^2(\frac{T}{2\pi}) \left(1 + O((\log T)^{-1/2+\epsilon})\right)$$

and $\mathcal{S}_2 \leq \mathcal{Q}_2 \log(\frac{T}{2\pi}) \left(1 + O((\log T)^{-1/2+\epsilon})\right)$. By Lemma 6 (iii)

$$\left| \frac{\mathcal{S}_1}{\mathcal{S}_2} \right| \geq (1/2) \log(\frac{T}{2\pi}) \frac{\mathcal{Q}_1}{\mathcal{Q}_2} (1 + o(1)) \geq \exp \left(\sqrt{\frac{\log M}{\log \log M}} (1 + o(1)) \right)$$

and thus we establish Theorem 2. □

It suffices to prove Lemma 7.

Proof of Lemma 7. We proceed to bound the various Σ_i . We begin with

$$\Sigma_i = \sum_{un \leq M} \frac{f(u)f(nu)(\log n)^i}{\sqrt{n}}$$

for $i = 1, 2$. We evaluate this by writing $(\log n)^i = \sum_{k|n} \Lambda_i(k)$. Inserting this expression we obtain

$$\begin{aligned} \Sigma_i &= \sum_{k \leq M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{nu \leq M/k \\ (nu, k)=1}} \frac{f(u)f(nu)}{\sqrt{n}} \leq \sum_{k \leq M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{n \leq M/k \\ (n, k)=1}} \frac{f(n)}{\sqrt{n}} \sum_{\substack{u \geq 1 \\ (u, kn)=1}} f(u)^2 \\ &\leq \sum_{k \leq M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{n \leq M/k \\ (n, k)=1}} \frac{f(n)}{\sqrt{n}} \prod_{(p, kn)=1} (1 + f(p)^2) = \mathcal{Q}_2 \sum_{k \leq M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}} \sum_{\substack{n \leq M/k \\ (n, k)=1}} \frac{f(n)}{\sqrt{ng(kn)}} \\ &\leq \mathcal{Q}_2 \sum_{k \leq M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}g(k)} \sum_{n=1}^{\infty} \frac{f(n)}{\sqrt{ng(n)}} = \mathcal{Q}_2 \prod_p \left(1 + \frac{f(p)}{\sqrt{pg(p)}}\right) \sum_{k \leq M} \frac{\Lambda_i(k)f(k)}{\sqrt{k}g(k)}. \end{aligned}$$

The expression in front of the last sum is clearly \mathcal{Q}_1 . Thus by Lemma 6 (iv)

$$\Sigma_1 \ll \mathcal{Q}_1 (\log T)^{1/2+\epsilon} \text{ and } \Sigma_2 \ll \mathcal{Q}_2 (\log T)^{1+\epsilon}.$$

Next note that $(\Lambda * \log)(r) \leq (\log r)^2$ and hence $\Sigma_3 \leq \Sigma_2 \ll (\log T)^{1+\epsilon}$. Next we estimate Σ_4 :

$$\begin{aligned} \Sigma_4 &= \sum_{\substack{a, b \leq M \\ (a, b)=1}} \frac{r_1(a, b)}{\sqrt{ab}} \sum_{g \leq \min(\frac{M}{a}, \frac{M}{b})} f(ag)f(bg) \\ &\leq \mathcal{Q}_1 \sum_{\substack{a, b \leq M \\ (a, b)=1}} \frac{f(a)f(b)|r_1(a, b)|}{\sqrt{ab}g(a)g(b)} \\ &\ll \mathcal{Q}_1 \left(\sum_{v \leq M} \frac{x_v}{\sqrt{vg(v)}} \right) \left(\sum_{a \leq M} \frac{\Lambda_2(a)f(a)}{\sqrt{ag(a)}} + \log T \sum_{a \leq M} \frac{\Lambda(a)f(a)}{\sqrt{ag(a)}} \right) \\ &\leq \prod_p \left(1 + \frac{f(p)}{\sqrt{p}} + f(p)^2 \right) (\log T)^{3/2+\epsilon} \end{aligned}$$

by Lemma 6 (iv). Finally we have

$$\Sigma_6 = \sum_{ur \leq M} \frac{\Lambda(r)f(u)f(ur)}{\sqrt{r}} = \sum_{r \leq M} \frac{\Lambda(r)f(r)}{\sqrt{r}} \sum_{\substack{u \leq M/r \\ (u, r)=1}} f(u)^2 \leq \prod_p (1+x_p^2) \sum_{r \leq M} \frac{\Lambda(r)f(r)}{\sqrt{rg(r)}}.$$

Once again by Lemma 6 (iv) we obtain $\Sigma_6 \ll \mathcal{Q}_2 \ll (\log T)^{1/2+\epsilon}$. \square

6. SMALL VALUES OF $\zeta'(\rho)$: PROOF OF THEOREM 3.

Proof of Theorem 3. We begin by noting that Theorem 3 is automatically true if there are infinitely many multiple zeros. Now assume that there are only finitely many multiple zeros of $\zeta(s)$. Suppose there exists a positive constant C' such that

for all $\gamma > C'$ all zeros of the zeta function are simple. We will now show that for each T sufficiently large that there exists a $\gamma \in [T, 2T]$ such that

$$|\zeta'(\rho)|^{-1} \geq \exp\left(c_5(1+o(1))\sqrt{\frac{\log T}{\log \log T}}\right) \quad (13)$$

and Theorem 3 follows. We now establish (13). Consider the sums

$$S_3 = \sum_{T_1 < \gamma < T_2} \zeta'(\rho)^{-1} |A(\rho)|^2 \text{ and } S_2 = \sum_{T_1 < \gamma < T_2} |A(\rho)|^2$$

where $A(s) = \sum_{k \leq M} x_k k^{-s}$ and x_k is an arbitrary real sequence. Here we choose T_1, T_2 such that

$$\zeta(\sigma + iT_j)^{-1} \ll T_j^\epsilon$$

where $T_1 = T + O(1)$ and $T_2 = 2T + O(1)$. This is possible by Theorem 14.16 of [17]. We shall establish:

Proposition 8. *Assume the Riemann hypothesis and that all but finitely many of the zeros of the Riemann zeta function are simple. If $\|\frac{x_n}{n}\|_1 \ll T^\epsilon$*

$$S_3 = \frac{T_2 - T_1}{2\pi} \sum_{hn \leq M} \frac{\mu(n)x_h x_{nh}}{nh} + O\left(T^\epsilon(M\|x_n\|_\infty + \|x_n\|_1 + T^{\frac{1}{2}}\|x_n^2\|_1^{\frac{1}{2}})\right)$$

for T sufficiently large.

Moreover by Proposition 4 we have

$$S_2 = (N(T_2) - N(T_1)) \sum_{m \leq M} \frac{x_m^2}{m} - \frac{T_2 - T_1}{\pi} \sum_{m \leq M} \frac{(\Lambda * x)(m)x_m}{m} + o(T)$$

respectively. We now choose $x_m = \sqrt{m}\mu(m)f(m)$ and suppose that $M < T^{2/3-10\epsilon}$. Note that $\|x_n\|_\infty \ll M^{\frac{1}{2}+\epsilon}$, $\|x_n\|_1 \ll M^{1+\epsilon}$ and thus

$$S_3 = \frac{T_2 - T_1}{2\pi} \left(\sum_{hn \leq M} \frac{f(h)f(nh)}{\sqrt{n}} + o(1) \right)$$

and

$$S_2 = (N(T_2) - N(T_1)) \sum_{m \leq M} f(m)^2 - \frac{T_2 - T_1}{\pi} \sum_{m \leq M} \frac{(\Lambda * f)(m)f(m)}{m} + o(T).$$

The second sum in S_2 is bounded by

$$\sum_{mp \leq M} \frac{(\log p)f(m)f(mp)}{\sqrt{p}} \ll \sum_{p \leq b} \frac{\log pf(p)}{\sqrt{p}} \sum_{\substack{m \leq M/p \\ (m,p)=1}} f(m)^2 \ll \left(\sum_{m \leq M} f(m)^2 \right) (\log T)^{1/2+\epsilon}.$$

With these observations in hand we obtain

$$\max_{T \leq \gamma \leq 2T} |\zeta'(\rho)|^{-1} \geq \log\left(\frac{T}{2\pi}\right)^{-1} \left(\frac{\sum_{hn \leq M} \frac{f(h)f(nh)}{\sqrt{n}}}{\sum_{m \leq M} f(m)^2} \right) (1 + o(1))$$

and by Soundararajan's calculation we obtain

$$\max_{T \leq \gamma \leq 2T} |\zeta'(\rho)|^{-1} \geq \exp\left((1+o(1))\sqrt{\frac{\log M}{\log \log M}}\right)$$

for $M < T^{\frac{2}{3}-10\epsilon}$ which yields (13).

It now suffices to establish Proposition 8.

Proof of Proposition 8. We consider the integral

$$I := \frac{1}{2\pi i} \int_{c+iT_1}^{c+iT_2} \zeta(s)^{-1} A(s) A(1-s) ds .$$

with $c = 1 + O((\log T)^{-1})$. Moving the contour left to the $1-c$ line yields $I = S_3 + H + I' + O(1)$ where

$$I' := \frac{1}{2\pi i} \int_{1-c+iT_1}^{1-c+iT_2} \zeta(s)^{-1} A(s) A(1-s) ds$$

and H are the horizontal contributions. We know from Proposition 4 that

$$I = \frac{T_2 - T_1}{2\pi} \sum_{nu \leq M} \frac{\mu(n)x_u x_{nu}}{nu} + O(M^\epsilon (\|x_n\|_\infty M + \|x_n\|_1)) .$$

Next we consider the contribution from the horizontal terms. We may verify that $|A(s)A(1-s)| \leq M \|\frac{x_n}{n}\|_1^2 + \|x_n\|_1 \|\frac{x_n}{n}\|_1$ for $1-c \leq \operatorname{Re}(s) \leq c$. Furthermore, since we have chosen the T_j such that $\zeta(\sigma + iT_j)^{-1} \ll T_j^\epsilon$, $H \ll T^\epsilon (M + \|x_n\|_1)$. We now consider the contribution of the left hand side. We have that $\zeta(s) = \chi(s)\zeta(1-s)$. Since $\chi(s) \asymp T^{1/2}$ and $\zeta(1-s) \asymp \log T$ for $\operatorname{Re}(s) = 1-c$ we have

$$\begin{aligned} I' &\ll (T^{1/2} \log T)^{-1} \|\frac{x_n}{n}\|_1 \int_{T_1}^{T_2} |A(1-c+it)| dt \\ &\ll (\log T)^{-1} \|\frac{x_n}{n}\|_1 \left(\int_{T_1}^{T_2} |A(1-c+it)|^2 dt \right)^{1/2} . \end{aligned}$$

The mean value theorem for Dirichlet polynomials asserts

$$\int_{T_1}^{T_2} \left| \sum_{n \leq N} \frac{a_n}{n^{it}} \right|^2 dt = (T_2 - T_1) \sum_{n \leq N} |a_n|^2 + O\left(\sum_{n \leq N} n |a_n|^2\right) .$$

Since $1-c = O((\log T)^{-1})$

$$\int_{T_1}^{T_2} |A(1-c+it)|^2 dt \ll T \sum_{n \leq M} x_n^2 + \sum_{n \leq M} \frac{x_n^2}{n} \ll T \|x_n^2\|_1 .$$

Thus we deduce that $I' \ll T^{\frac{1}{2}} (\log T)^{-1} \|\frac{x_n}{n}\|_1 \|x_n^2\|_1^{\frac{1}{2}}$. Collecting estimates yields Proposition 8. \square

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